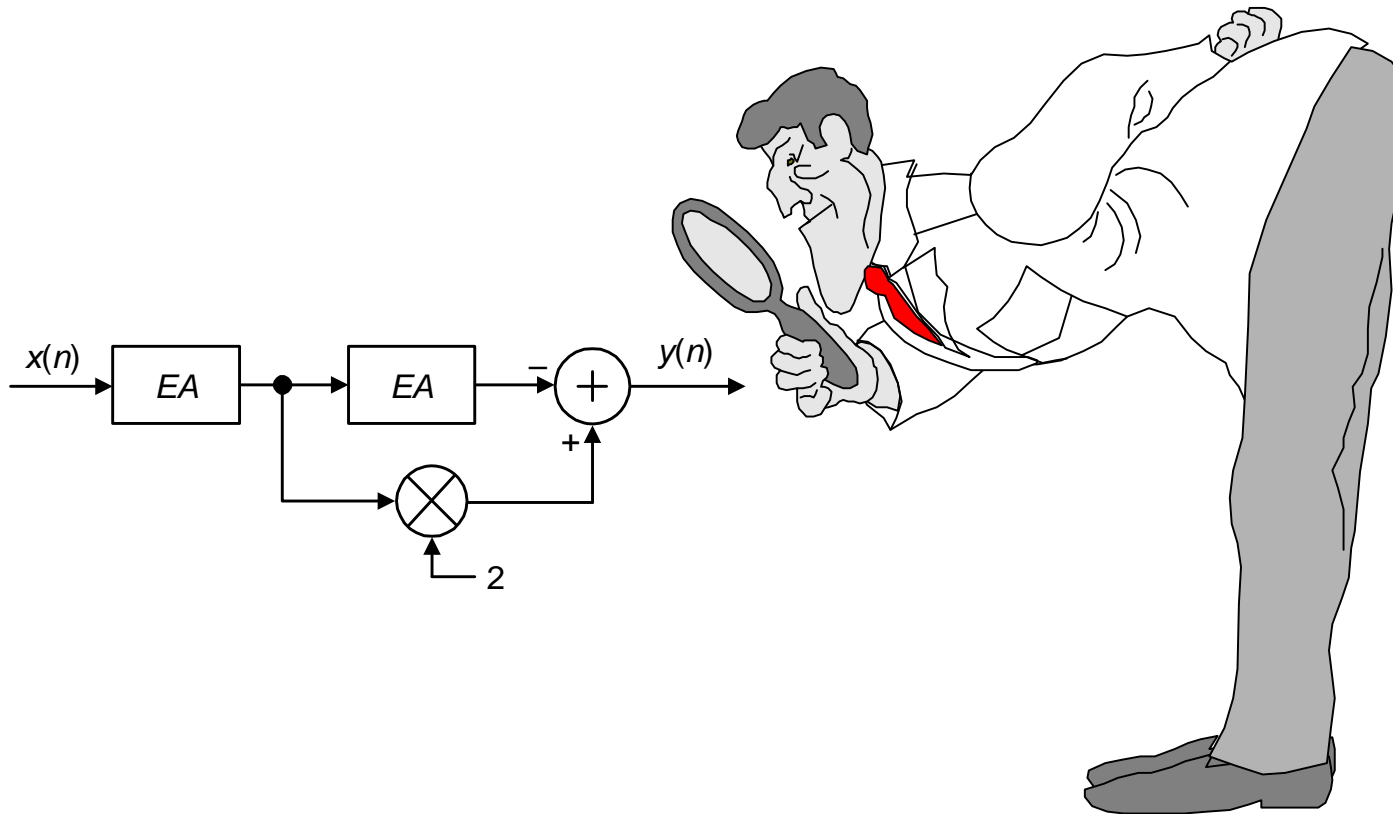# Reduced-Delay Data Smoothing

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Low-Delay Moving Averagers

- **There was a recent post on the comp.dsp newsgroup that mentioned a process called a "Hull Moving Average" [1].**

- **Being unfamiliar with this process, I started searching the Internet for information on the Hull Moving Average (Hull-MA) .**

- **The Hull-MA is a moving average operation used by Stocks and Commodities investors (active traders) in their Technical Analysis (TA) activities.**

    - **Proposed by Alan Hull [2]**

- **Technical Analysts combine moving averages with price movements to trigger their buy, or sell, decisions.**

    - **However, <u>they prefer moving average curves with minimum delay</u>!**
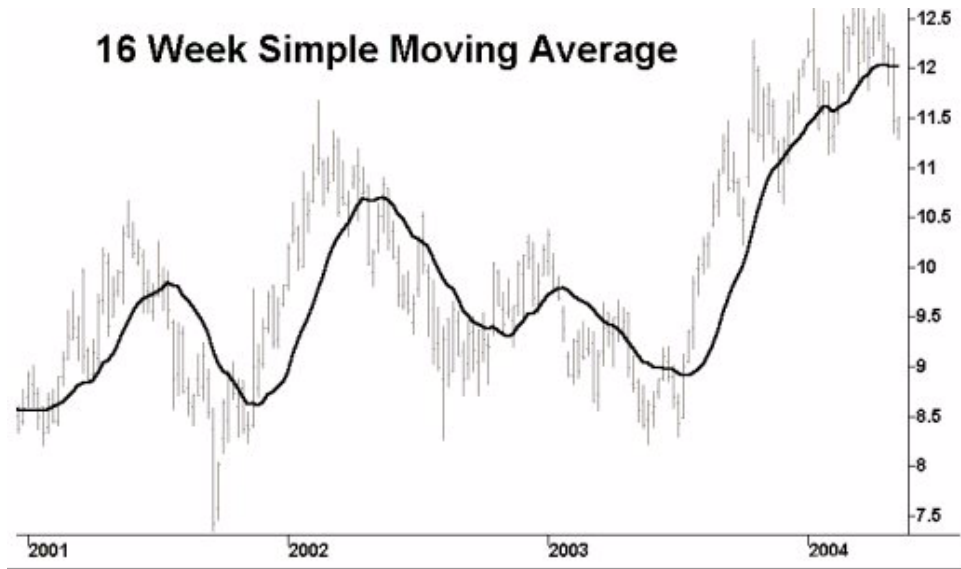
Notice the considerable delay.
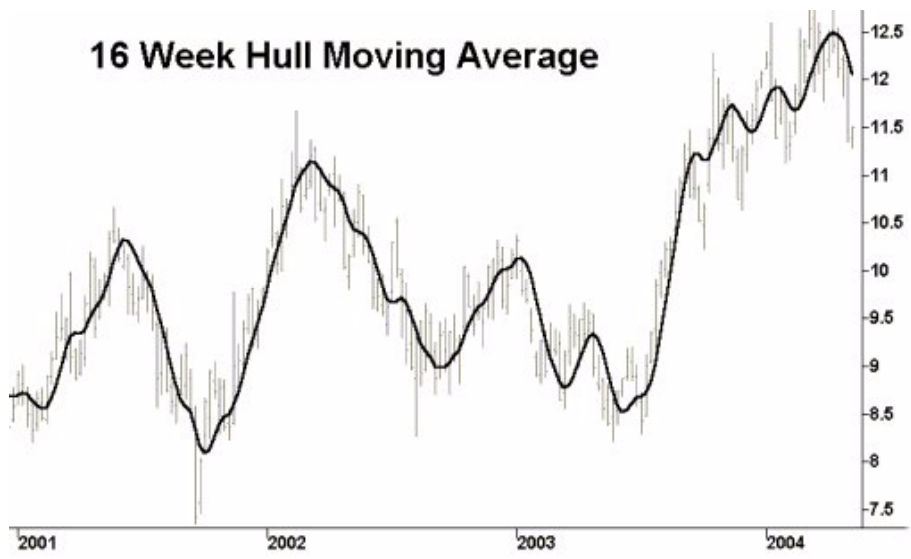
Desirable reduced delay.

Figures copied from Reference [2] web site.

● **Is investing in stocks or commodities the same as gambling?**

● **Not according to Blackie Sherrode [3]:**

- **"If you bet on a horse, that's gambling. If you bet you can make three spades, that's entertainment. If you bet cotton will go up three points, that's business. See the difference?"**

- **The purpose of the Hull-MA is to provide:**
    - **noise reduction (smoothing, lowpass filtering),**
    - **with a <u>delay that's less than that of a standard moving averager</u>.**

- **The Hull-MA is a parallel/cascade arrangement of simple tapped-delay line moving average subfilters as in Figure 1(a).**
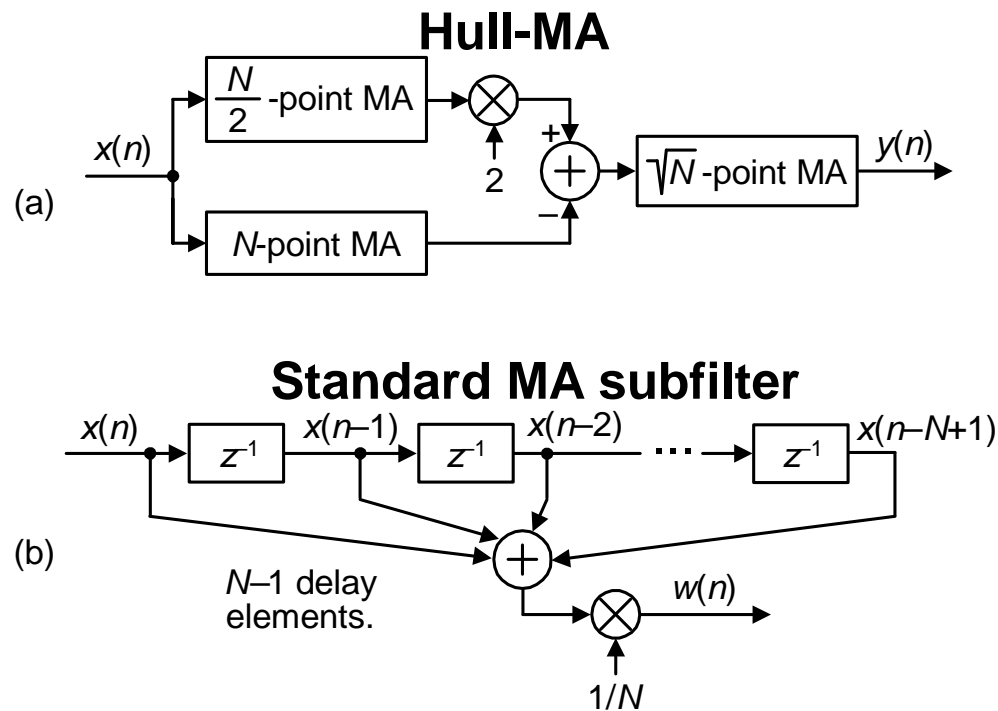   - **Uses the simple tapped-delay line moving average (MA) subfilters shown in Figure 1(b).**

Hull-MA

(a)

Standard MA subfilter

(b)

<u>**Figure 1**</u>

- **The MA subfilters are *N*-point, *N*/2-point, and sqrt(*N*)-point.**

    - *N* **is an even integer.**
    - **When sqrt(*N*) is not an integer, then we round sqrt(*N*) to the nearest integer.**

- **In my Internet searching I also encountered another low-delay MA scheme:**

    - **called the "triple exponential averager" (TEMA) scheme shown in Figure 2(a).**
    - **The TEMA process uses simple exponential averagers (EAs), one of which is shown in Figure 2(b).**
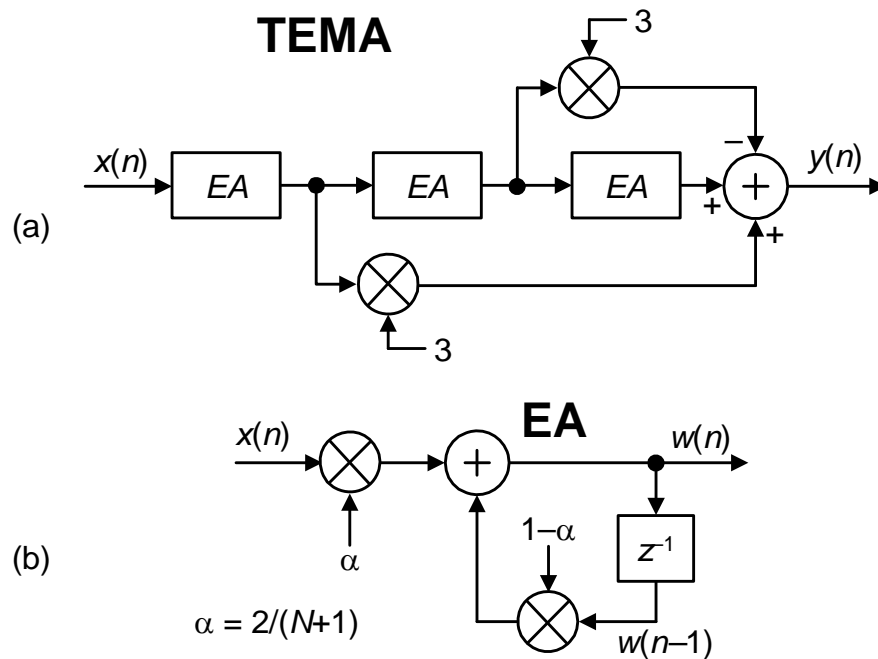   - **In the DSP community, the EA subfilters are also called "leaky integrators."**

Figure 2

● **The α weighting constant in an EA subfilter, defined for an *N*-point TEMA process, is:**

$$\alpha = \frac{2}{N+1}.$$

  - **For example, for an *N*=16-point TEMA process, α = 2/(16+1) = 0.1176.**

● **The TEMA scheme was introduced to the stock market world by Patrick Mulloy [4,5],**

  - **based on an original idea by John Tukey [6], inventor of the radix-2 FFT!**

● **Looking at the HULL-MA and TEMA smoothing schemes, I recalled the digital filter design scheme known as "filter sharpening"**

  - **described by comp.dsp-er Matt Donadio [7], and mathematician Richard Hamming (of Hamming window fame) [8].**

● **Filter sharpening, also originally proposed by Tukey, is a method of building *improved* filtering systems using parallel and cascade combinations of subfilters,**

  - **similar to the HULL-MA and TEMA data smoothing schemes.**

- **John Tukey**

John Tukey
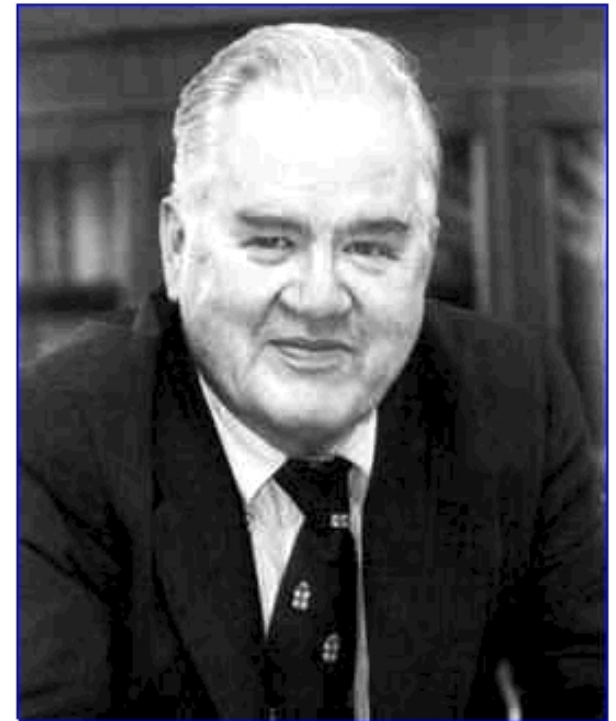
- **On business trips, Tukey always took his paddle along because he loved playing table tennis.**

8

● **Reference [8] also describes a filter sharpening scheme, called "Twicing" by John Tukey, shown in Figure 3.**

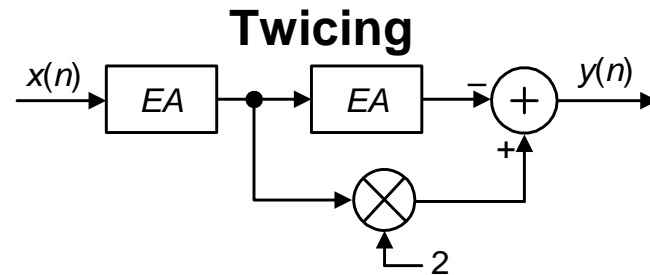 - **The Twicing process also uses the EA subfilters from Figure 2(b).**

Twicing

Figure 3

● **I've studied the HULL-MA, TEMA, and Twicing data smoothing schemes,**

  - **and found the TEMA process has no discernable advantage over the Twicing process for data smoothing.**

● **Given that, let's look at the HULL-MA and Twicing reduced-delay data smoothing schemes in more detail**

  - **(much more detail than the stock market people do).**

● **Figure 4 shows the impulse responses of three moving averagers,**

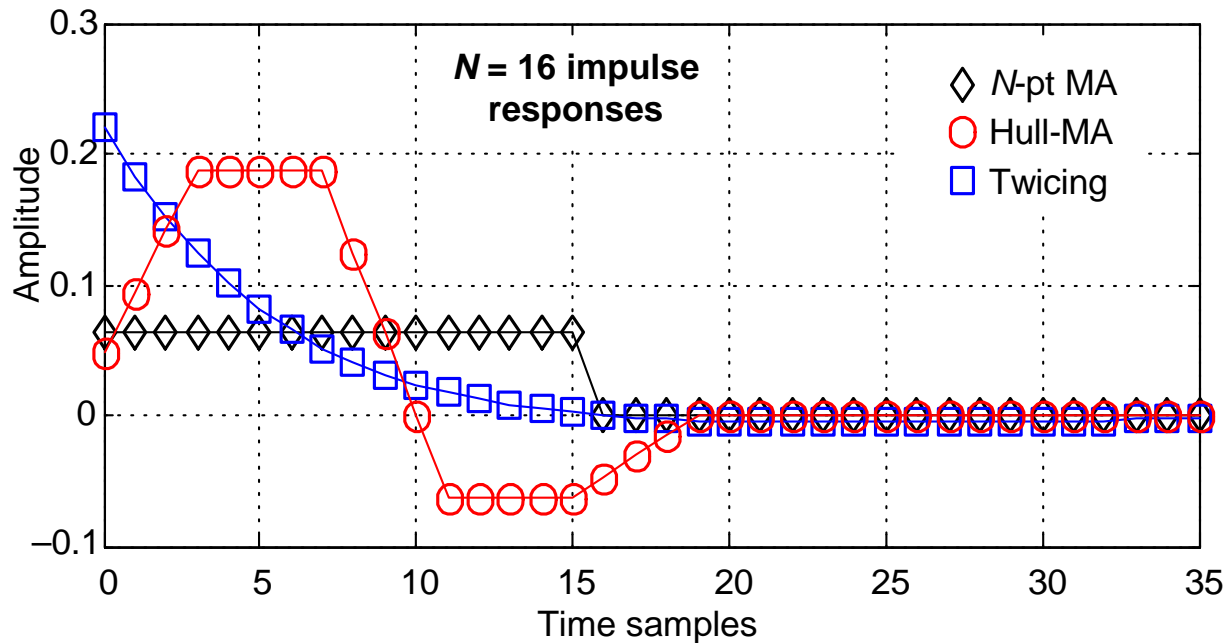 - **where only the standard *N*-point MA has a symmetrical imp. response.**

Figure 4

● **The impulse responses of standard MA subfilters never go negative, but**

  - **this is not so for the Hull-MA.**

  - **That's because the imp. response of the *N*-point MA subfilter is longer than the *N*/2-point MA subfilter's imp. response.**

- **Figure 5 shows the frequency magnitude responses of four moving averagers.**

N = 16 frequency mag. responses

Legend:
- *N*-pt MA (dotted)
- Single EA (dash-dot)
- Hull-MA (red dashed)
- Twicing (blue solid)

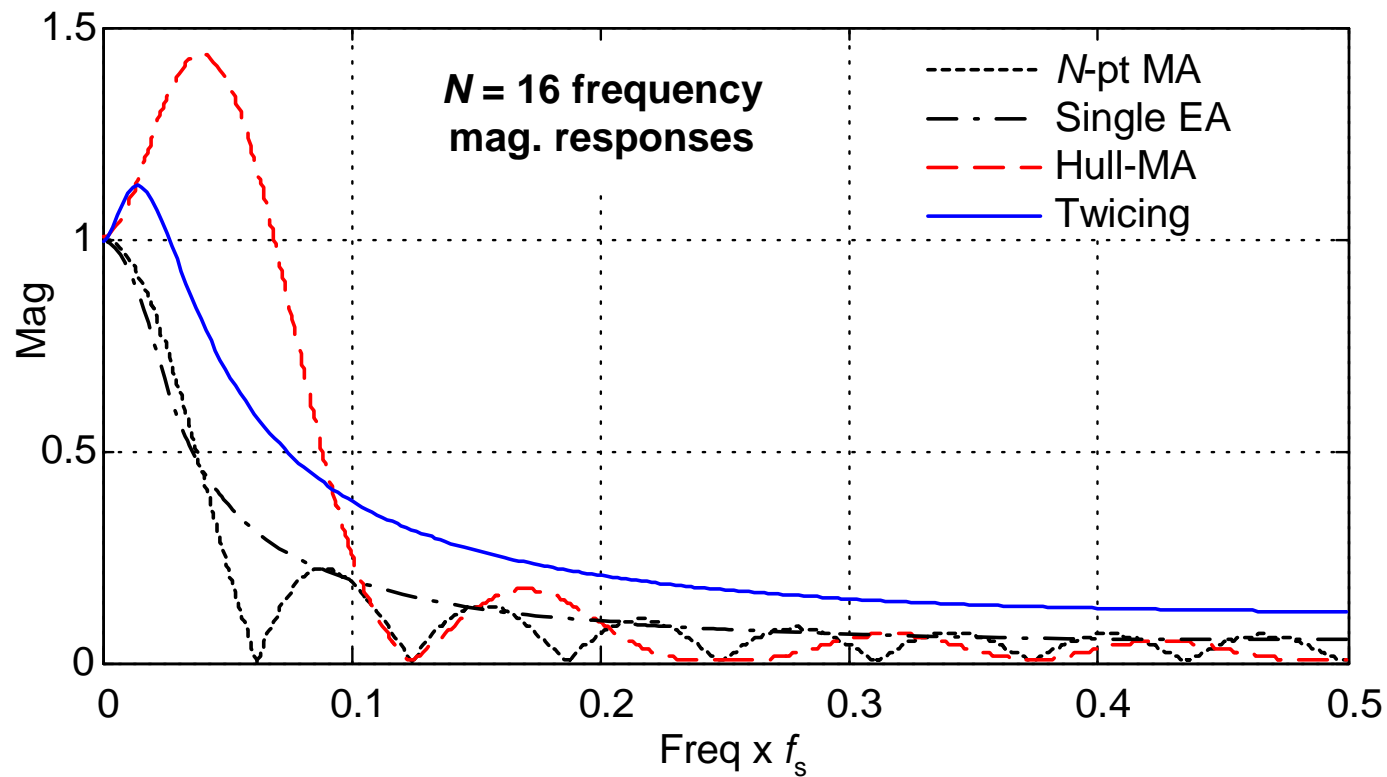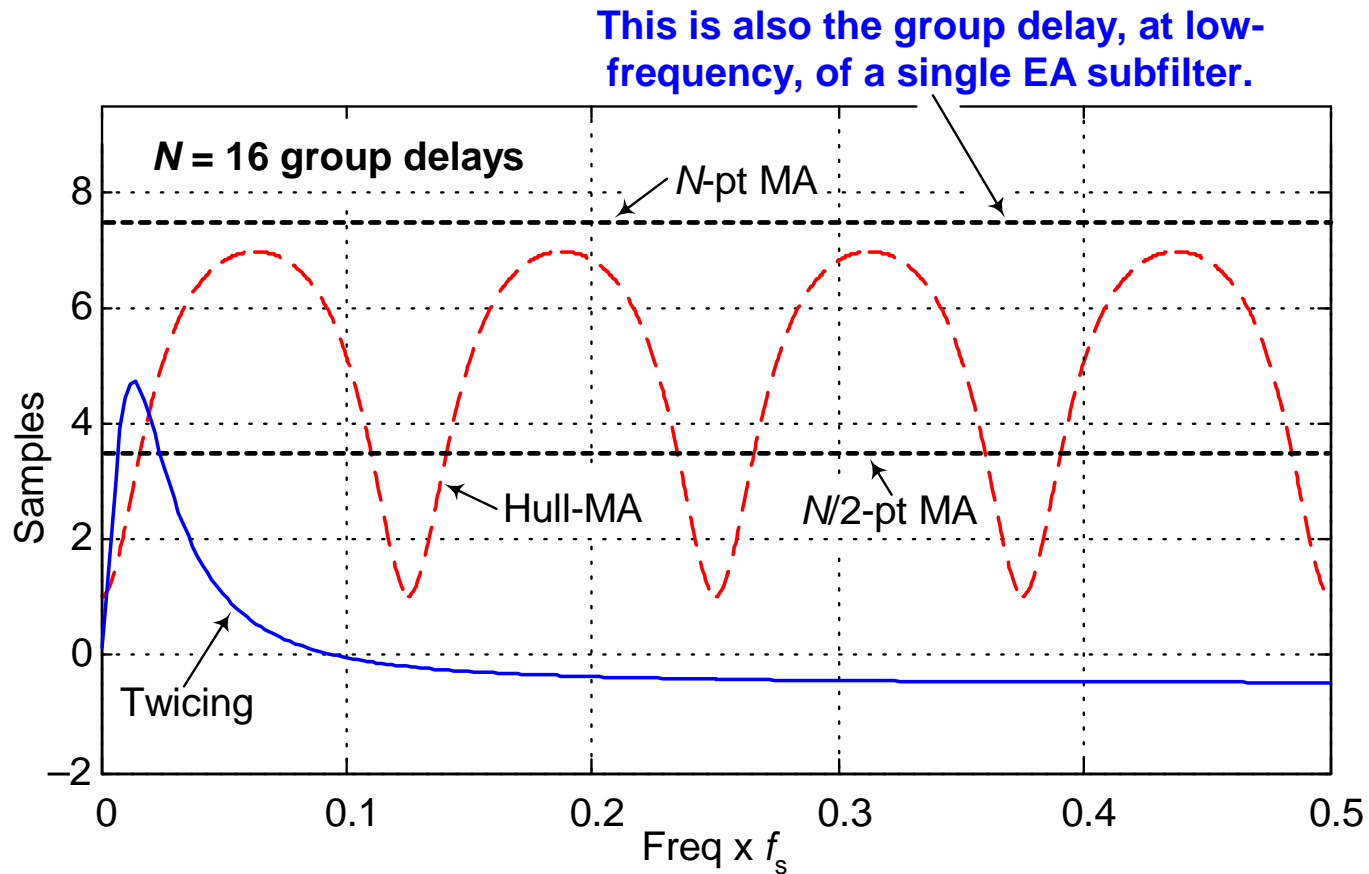X-axis: Freq x f_s
Y-axis: Mag

Figure 5

- **All are lowpass filters.**

- **The Hull-MA has better stopband attenuation (data smoothing) than Twicing.**

- **Figure 6 shows the group delays of five moving averagers.**
  - **For completeness we included an *N*/2-point MA group delay line in Figure 6.**

Figure 6

- **<u>Delay of the Hull-MA filter is less than the delay of an *N*-point MA subfilter!</u>**

- **<u>Delay of the Twicing filter is less than the delay of a single EA subfilter!</u>**

● **The Twicing filter has the smallest average group delay,**

   - **and that is what we want.**

● **The following table gives the group delays (measured in samples) versus N.**

   - **Those group delay values are averages over the full frequency range,**

   - **except for the Twicing values.**

N	N-point MA	$N/2$-point MA	Hull-MA	Twicing*
16	7.5	3.5	5.0364	2.786
36	17.5	8.5	10.689	3.757
64	31.5	15.5	20.046	4.129
100	49.5	24.5	29.596	4.305

* Over the frequency range of 0 -to- $f_s/20$ Hz.

● **Note that the values for N are such that N, $N/2$, and sqrt(N) are all integers.**

   - **That's the condition that minimizes the average Hull-MA group delay.**

● **Let's look at our averagers in action, in Figure 7, for $N = 16$.**

  - **Value Δ is the delay of an input peak to a standard N-point MA's peak.**
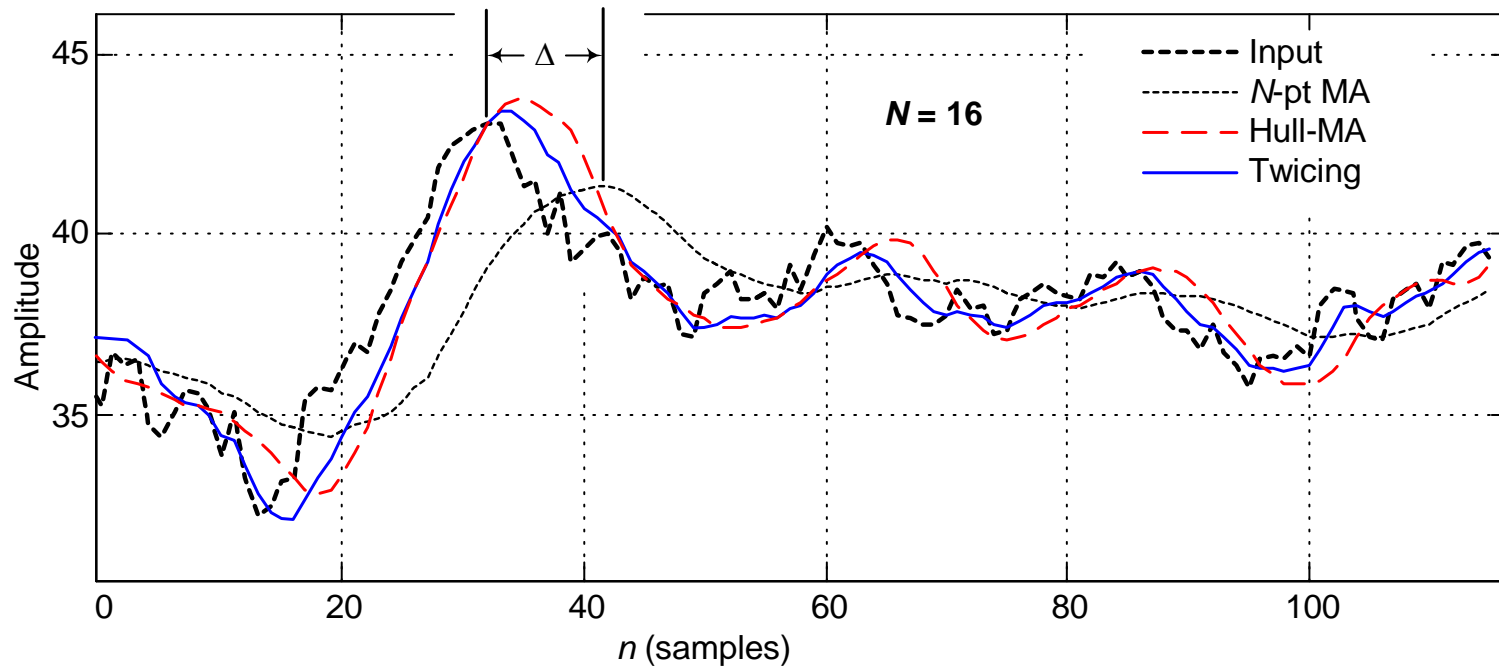

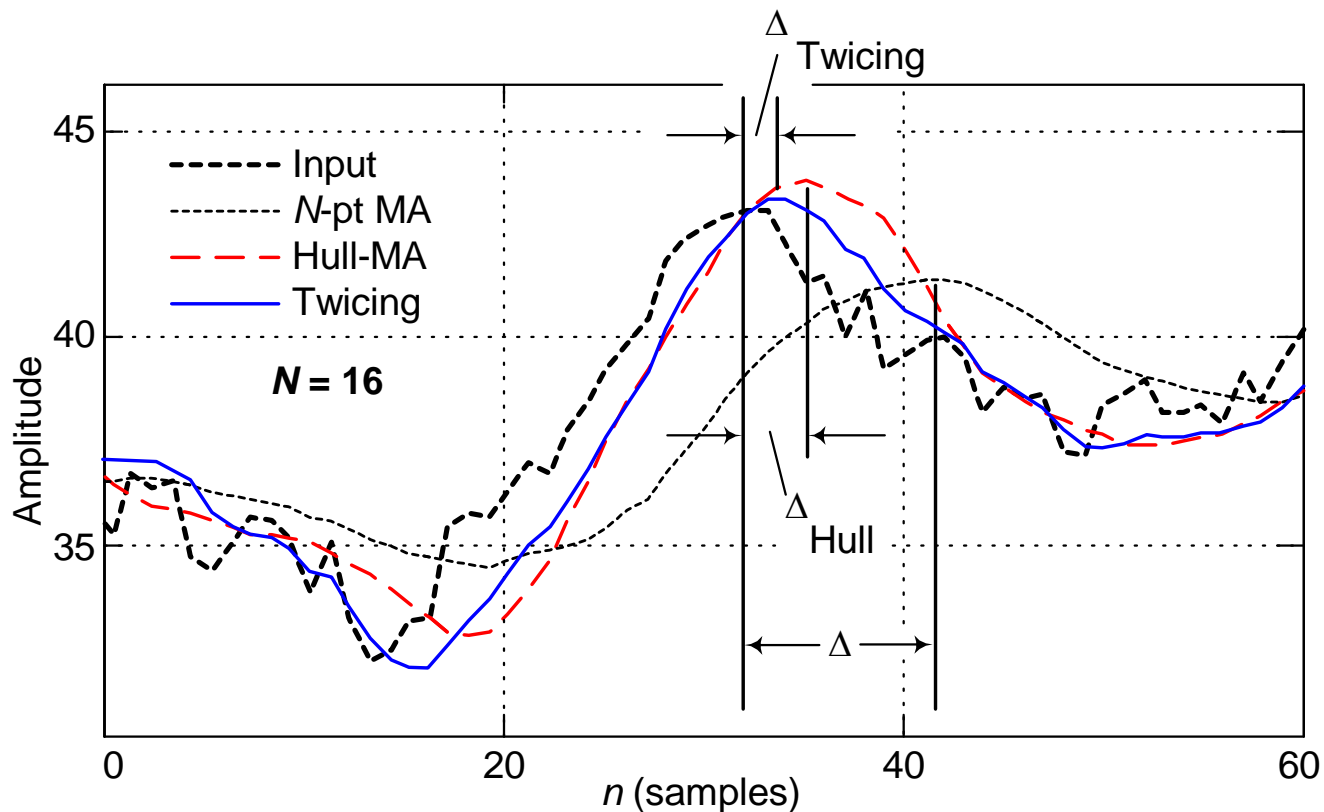**Figure 7**

● **Zooming in, we see:**

Figure 8

● **The Hull-MA has improved smoothing,**

    - **because its stopband attenuation was better than the Twicing scheme in Figure 5.**

● **However, the Twicing process has the minimum delay, $\Delta_{\text{Twicing}} < \Delta_{\text{Hull}}$, as predicted in Figure 6.**

● **The Hull-MA and Twicing processes in Figure 1(a) and Figure 3(a) are for "offline", or block, processing, where the outputs of all subfilters are available.**

  - **For real-time processing we need to <u>insert delay lines</u> to time-synchronize the outputs of various subfilters.**

● **For real-time processing, a Hull-MA is implemented as shown in Figure 9(a),**

  - **where we've eliminated the 'multiply-by-two' operation by doubling the coefficient values in the $N/2$-point MA.**
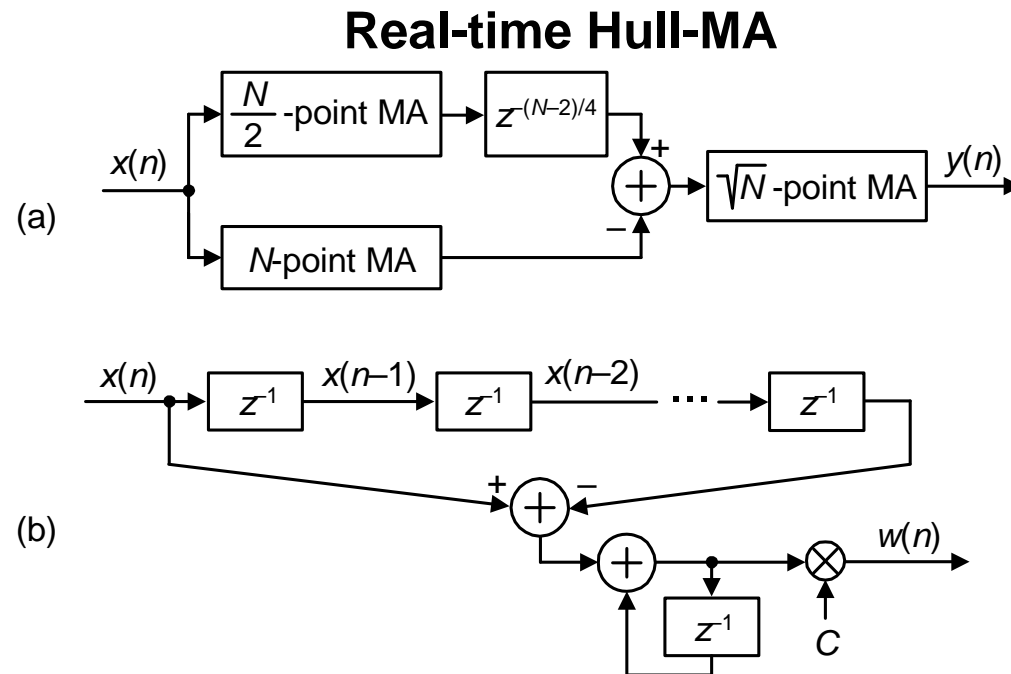
Real-time Hull-MA

<u>Figure 9</u>

● **The inserted-delay line length in Figure 9(a) is the group delay of an
   N/2-point MA, i.e., (N–2)/4 samples.**

● **The number of additions per output sample of the Hull-MA subfilters can
   be drastically reduced**

   - **by using the recursive-MA structure shown in Figure 9(b).**

● **The multiplying coefficient, *C*, in Figure 9(b) is defined by:**

$$C = \begin{cases} \dfrac{1}{N}, & \text{for } N\text{-point MA subfilter} \\ \dfrac{4}{N}, & \text{for } N/2\text{-point MA subfilter} \\ \dfrac{1}{round(\sqrt{N})}, & \text{for sqrt}(N)\text{-point MA subfilter} \end{cases}$$

● **For real-time processing, the Twicing scheme must be implemented as shown in Figure 10 with a Delay line preceding the multiply-by-two operation.**
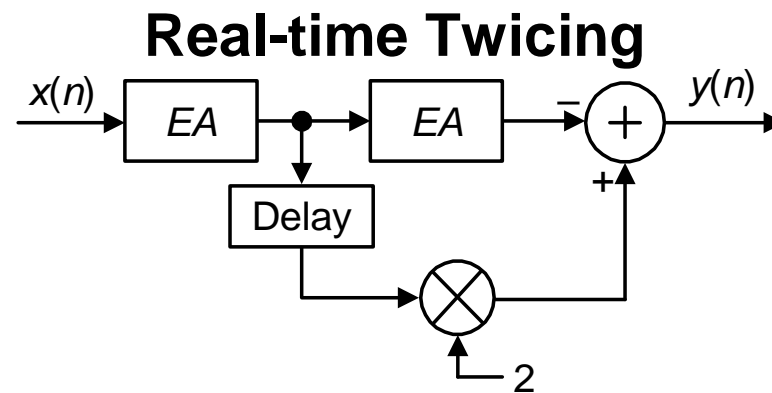
Real-time Twicing

Figure 10

● **The question is: How many unit-delay elements are in the Figure 10 "Delay" line?**

- **The group delay of a general EA filter takes the form shown in Figure 11. We want to know the group delay at zero Hz (DC).**

- **The group delay, at low frequencies, of our EA is $D = (N-1)/2$ samples, as derived in the Appendix, leading us to the Figure 12 implementation.**
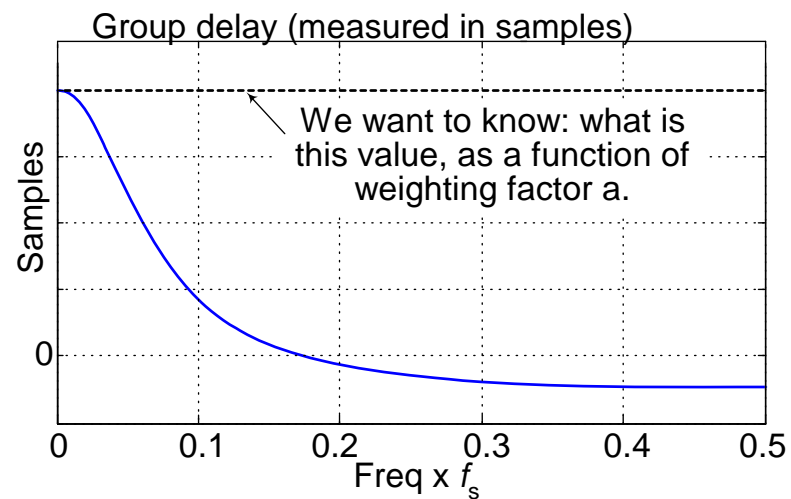
Group delay (measured in samples)

We want to know: what is this value, as a function of weighting factor a.

Figure 11

Real-time Twicing

$x(n)$ → EA → EA → $+$ → $y(n)$

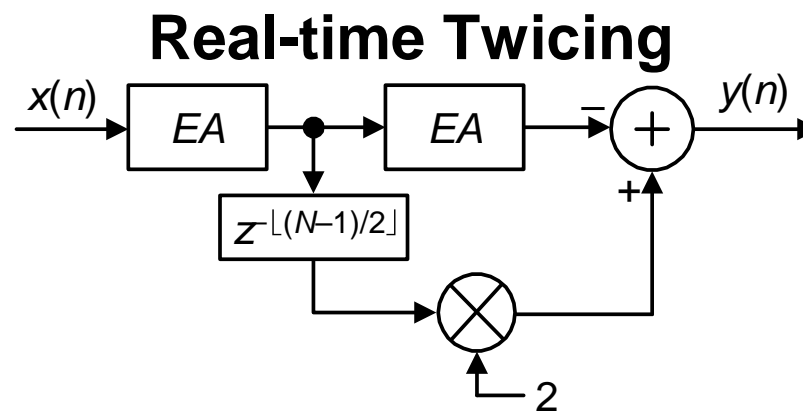$z^{-\lfloor (N-1)/2 \rfloor}$

$\otimes$ — 2

Figure 12

- **To keep it integer-valued, the delay line length in Figure 12 is $\lfloor D \rfloor = \lfloor (N-1)/2 \rfloor$,**

    - **where $\lfloor D \rfloor$ means the integer part of D.**

● **Of course, we can eliminate the single 'multiply-by-two' operation in Figure 12 using the structure shown in Figure 13.**
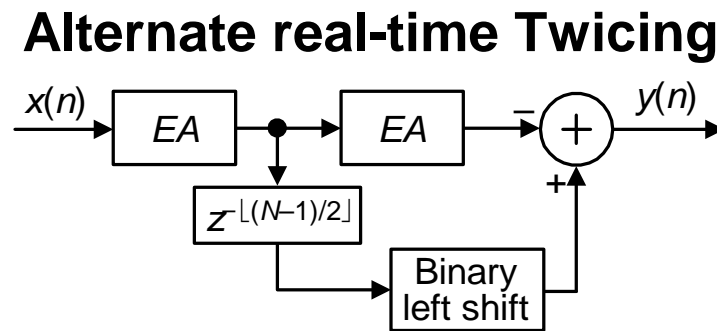
Alternate real-time Twicing

Figure 13

● **To further reduce the Twicing computational workload, we can use the "single multiplier" EA subfilter structure given in Figure 14(b),**

   - **introduced to me by Vladimir Vassilevsky (our intelligent, gracious, and good-looking host).**
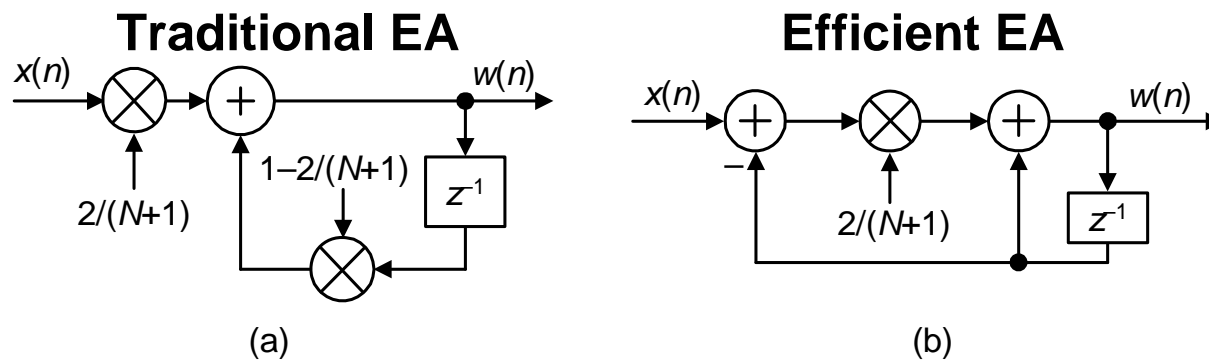

Figure 14

● **In our relentless quest for computational efficiency, know that**

   - **reference [9] gives methods for implementing exponential averagers (EAs) <u>without using multipliers at all</u>.**

CONCLUSION:

- **We've introduced two simple data smoothing schemes: Hull-MA and Twicing,**

    - **that have reduced delay beyond traditional moving average methods.**

- **We also showed computationally-efficient implementations of the reduced-delay data smoothing schemes**

- **WARNING: for proper operation,**

    - **all the subfilters in the Hull-MA and Twicing structures must have unity gain at DC (zero Hz).**

- **For any given DSP application requiring reduced-delay data smoothing,**

    - **I suggest implementing both the Hull-MA and Twicing methods to see which one appears to work the best.**

REFERENCES:

[1] comp.dsp Newsgroup, From: Jessica, Subject: Data Smoothing Filter, Date: Feb. 1, 2010.

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[3] A. Tobias, *The Only Investment Guide You'll Ever Need*, Harcourt, Brace, Jovanovich Pub. Co., 1977, pp. 108.

[4] P. Mulloy, "Smoothing Data With Less Lag", *Technical Analysis of Stocks & Commodities* magazine, Vol. 12, February 1994.

[5] http://traderscom.stores.yahoo.net/-v12-c02-smoothi-pdf.html

[6] J. Tukey, *Exploratory Data Analysis*, Addison-Wesley Pub. Co., Reading, Massachusetts, Chapter 16, 1977.

[7] M. Donadio, "Lost Knowledge Refound: Sharpened FIR Filters", *IEEE Signal Processing Magazine*, Vol. 20, No. 5, Sept. 2003, pp. 61-63. Also found in Chapter 1 of the book "Streamlining Digital Signal Processing", IEEE Press/Wiley.

[8] R. Hamming, *Digital Filters*, 3rd Ed., Dover Publications, Mineola, NY, 2004, pp. 140-145.

[9] R. Lyons, "Multiplierless Exponential Averaging", http://www.dsprelated.com/showarticle/72.php

[10] R. Lyons, *Understanding Digital Signal Processing*, 2nd Ed., Prentice Hall, Upper Saddle River, New Jersey, 2004, pp. 436-437.

APPENDIX

Here's how we estimate the low-frequency group delay of the Figure 2(b) exponential averager (EA). The group delay of a general EA filter takes the form shown in Figure 11. We want to know the group delay at zero Hz (DC).

The frequency response of an EA, whose weighting factor is α, is:

$$H_{\exp}(\omega) = \frac{\alpha}{1-(1-\alpha)e^{-j\omega}} = \frac{\alpha}{1-(1-\alpha)\cos(\omega) + j(1-\alpha)\sin(\omega)} \qquad (A-1)$$

where $-\pi \leq \omega \leq \pi$ radians/sample [10]. The phase response of the averager is:

$$\phi_{\exp}(\omega) = \tan^{-1}\left(\frac{-(1-\alpha)\cdot\sin(\omega)}{1-(1-\alpha)\cdot\cos(\omega)}\right). \tag{A-2}$$

The group delay of the averager is the negative of the derivative of $\phi_{\exp}(\omega)$. Taking the derivative of Eq. (A–2), we have the truly messy:

$$\frac{d[\phi_{\exp}(\omega)]}{d\omega} = \frac{\dfrac{-(1-\alpha)\cdot\cos(\omega)}{1-(1-\alpha)\cdot\cos(\omega)} + \dfrac{(1-\alpha)^2\cdot\sin^2(\omega)}{1-2(1-\alpha)\cdot\cos(\omega)+(1-\alpha)^2\cdot\cos^2(\omega)}}{1 + \dfrac{(1-\alpha)^2\cdot\sin^2(\omega)}{1-2(1-\alpha)\cdot\cos(\omega)+(1-\alpha)^2\cdot\cos^2(\omega)}}. \tag{A-3}$$

We could simplify Eq. (A–3), but here we don't have to. Instead, we merely evaluate Eq. (A–3) at $\omega = 0$ to estimate the exponential averager's low-frequency group delay. Doing that we have the sweet and simple expression:

$$\left.\frac{d[\phi_{\exp}(\omega)]}{d\omega}\right|_{\omega=0} = \frac{\alpha-1}{\alpha}. \tag{A-4}$$

Next we negate Eq. (A–4), set $\alpha = 2/(N+1)$, and state that the low-frequency group delay, D, of a single EA subfilter for our N-point Twicing application is approximately:

$$D = \left.\frac{-d[\phi_{\exp}(\omega)]}{d\omega}\right|_{\omega=0} = \frac{N-1}{2} \text{ samples.} \tag{A-5}$$